

Chiral Multiplets in $N = 1$ Dual String PairsGottfried Curio¹*Humboldt-Universität zu Berlin, Institut für Physik, D-10115 Berlin, Germany*

We compare the spectrum of chiral multiplets in the $N = 1$ vacua of the heterotic string on a Calabi-Yau together with an $E_8 \times E_8$ vector bundle and F -theory on a smooth Calabi-Yau fourfold. Under suitable restrictions we show agreement using an index-computation.

¹email: curio@qft2.physik.hu-berlin.de

Two of the most interesting classes of models coming from string theory compactifications with $N = 1$ supersymmetry in four dimensions are obtained from the heterotic string on a three-dimensional Calabi-Yau Z with a vector bundle V respectively from F -theory on an elliptically fibered Calabi-Yau fourfold $X^4 \rightarrow B^3$. These cases are of even greater interest in view of the duality philosophy that one should get dual models by adiabatically extending the eightdimensional duality between the heterotic string on T^2 and F -theory on $K3$ [1]. So we will furthermore assume that the heterotic Calabi-Yau is elliptically fibered $Z \rightarrow B^2$ and that the F -theory fourfold Calabi-Yau is $K3$ fibered $X^4 \rightarrow B^2$. This involves also the existence of a P^1 fibration $B^3 \rightarrow B^2$.

In this note we will show a duality matching among the concerned moduli assuming that we are in the case without an unbroken gauge group. So we will assume that V is an $E_8 \times E_8$ vector bundle and that X^4 is a smooth Weierstrass model; for worked out examples cf. [12]. The comparison works technically similar to the matching [2] of the number $\chi/24$ of spacetime-filling threebranes [3] occuring in the course of tadpole-cancellation to the number h of fivebranes (wrapping the fibre F of the elliptic fibration $Z \rightarrow B^2$) occuring in the context of the heterotic string anomaly cancellation

$$\lambda(V_1) + \lambda(V_2) + h[F] = c_2(Z)$$

in connection with a $E_8 \times E_8$ vector bundle $V = V_1 \times V_2$. Note also that at the end of this paper we remark on circumstances that the actual computation carried out here will find its proper place when embedded in an enlarged context.

So we consider F -theory on a smooth elliptically fibered Calabi-Yau fourfold X^4 with base B^3 which can be represented by a smooth Weierstrass model. We assume even more that the rank v of the gauge group in four dimensions is zero (,i.e. also no $U(1)$ factors). Now one can expect that the spectrum would get contributions from Kaehler and complex structure parameters related to $h^{1,1} - 2$ (not counting the unphysical F -theory elliptic fibre as well as not counting the class corresponding to the heterotic dilaton) and $h^{3,1}$ respectively as well as from $h^{2,1}$ giving in total $h^{1,1} - 2 + h^{2,1} + h^{3,1}$ parameters. One can see from a M -theory versus F -theory consideration that these contributions divide themselves in $4D$ between chiral and vector multiplets according to whether or not they come from the threefold base B^3 of the F -theory elliptic fibration (just as in the analogous $6D$ $N = 2$ case [4]; we will assume here $v = 0$ anyway).

So following this line of reasoning one would expect for the rank v of the $N = 1$ vector multiplets (unspecified hodge numbers relate to X^4) (cf. [6],[11],[12])

$$v = h^{1,1} - h^{1,1}(B^3) - 1 + h^{2,1}(B^3)$$

and for the number c of $N = 1$ neutral chiral (resp. anti-chiral) multiplets

$$\begin{aligned} c &= h^{1,1}(B^3) - 1 + h^{2,1} - h^{2,1}(B^3) + h^{3,1} \\ &= h^{1,1} - 2 + h^{2,1} + h^{3,1} - v \end{aligned}$$

Note that we are speaking here of the generic gauge group and not of the situation where one tunes the deformations of the Calabi-Yau to unhiggs an unbroken gauge group and correspondingly counts only the number of deformations *preserving* a particular singular locus (cf. [14]).

Now one has [3]

$$\frac{\chi}{6} - 8 = h^{1,1} - h^{2,1} + h^{3,1}$$

so that one finds

$$c = \frac{\chi}{6} - 10 + 2h^{2,1} - v$$

Now one can compute the Euler number of X in terms of topological data of the base B^3 of the elliptic fibration according to [3]

$$\frac{\chi}{24} = 12 + 15 \int_{B^3} c_1^3(B^3)$$

One can go even further: assuming, in the light of the application to duality with the heterotic string, that X^4 is a $K3$ fibration over a twofold base B^2 (so that B^3 is a P^1 fibration over B^2) one gets [2]

$$\frac{\chi}{24} = 12 + 90 \int_{B^2} c_1^2(B^2) + 30 \int_{B^2} t^2$$

where t encodes the P^1 fibration structure [2]: assume the P^1 bundle over B^2 given by the projectivization of a vector bundle $Y = \mathcal{O} \oplus \mathcal{T}$, with \mathcal{T} a line bundle over B^2 ; then $t = c_1(\mathcal{T})$. So one gets in the case with no unbroken gauge group

$$c_F = \frac{\chi}{6} - 10 + 2h^{2,1} = 38 + 360 \int_{B^2} c_1^2(B^2) + 120 \int_{B^2} t^2 + 2h^{2,1}$$

On the other hand we have to count the moduli on the heterotic side. Here one has contributions $m_{geo} = h^{1,1}(Z) + h^{2,1}(Z)$ from geometrical moduli plus the bundle moduli. Concerning the former let us assume that the dual heterotic Calabi-Yau threefold Z is smooth so that [7]

$$\chi(Z) = -60 \int_{B^2} c_1^2(B^2)$$

and assume furthermore that its smooth Weierstrass model is general [5], i.e. has only one section (a typical counterexample being the $CY^{19,19} = B_9 \times_{P^1} B_9$ with the del Pezzo

B_9 , the blow up of P^2 in the nine points of intersection of two cubics, cf. [10], [11]). So $h^{1,1}(Z) = h^{1,1}(B^2) + 1$; this gives using Noethers formula that

$$h^{1,1}(Z) = \int_{B^2} c_2(B^2) - 1 = 11 - \int_{B^2} c_1(B^2)^2$$

So one gets for the number of geometrical moduli

$$\begin{aligned} m_{geo} &= h^{1,1}(Z) + h^{2,1}(Z) = 2h^{1,1}(Z) - \frac{\chi}{2} \\ &= 22 + 28 \int_{B^2} c_1^2(B^2) \end{aligned}$$

Concerning the contribution of the bundle moduli of the $E_8 \times E_8$ bundle V on Z let us recall the setup of the index-computation in [2]. As the usual quantity suitable for index-computation $\sum_{i=0}^3 (-1)^i h^i(Z, V)$ vanishes by Serre duality one has to introduce a further twist and to compute a character-valued index. Now because of the elliptic fibration structure one has on Z the involution τ coming from the "sign-flip" in the fibers which we furthermore assume has been lifted to an action on the bundle. The character-valued index

$$I = -\frac{1}{2} \sum_{i=0}^3 (-1)^i \text{Tr}_{H^i(Z, V)} \tau$$

simplifies by the vanishing of the ordinary index to

$$I = -\sum_{i=0}^3 (-1)^i h^i(Z, V)_e$$

where the subscript "e" (resp. "o") indicates the even (resp. odd) part. As we have the gauge group completely broken one finds

$$I = n_e - n_o$$

denoting by $n_{e/o}$ the number $h^1(Z, V)_{e/o}$ of massless even/odd chiral superfields. Now one gets with Wittens index formula [8],[12]

$$I = 16 + 332 \int_{B^2} c_1^2(B^2) + 120 \int_{B^2} t^2$$

that one has for the number of the bundle moduli $m_{bun} = n_e + n_o = I + 2n_o$

$$m_{bun} = 16 + 332 \int_{B^2} c_1^2(B^2) + 120 \int_{B^2} t^2 + 2n_o$$

So adding up one gets in total

$$\begin{aligned} c_{het} &= h^{1,1}(Z) + h^{2,1}(Z) + I + 2n_o \\ &= 38 + 360 \int_{B^2} c_1^2(B^2) + 120 \int_{B^2} t^2 + 2n_o \end{aligned}$$

Now on the F -theory side the modes odd under the involution τ' corresponding to the heterotic involution τ correspond to the $h^{2,1}(X^4)$ classes [2]. Let us assume that no 4-flux was turned on (which is not a free decision [9] in general); otherwise there are further twistings possible (cf. [2], sect. 4.4, and also [13]) which account for a possible multi-component structure of the bundle moduli space (cf. also [14] for the case of $SU(n)$ bundles). This clearly deserves further study to embed the simple picture employed here in the more general context.

So one gets complete matching with $n_o = h^{2,1}(X^4)$.

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